ECS 452: Digital Communication Systems **2013/1**

HW 1 — Due: July 24

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider the **two-path channels** in which the receive signal is given by

$$y(t) = \beta_1 x(t - \Delta t_1) + \beta_2 x(t - \Delta t_2).$$

Four different cases are considered.

(a) Small $|\Delta t_1 - \Delta t_2|$ and $|\beta_1| \gg |\beta_2|$

(b) Large $|\Delta t_1 - \Delta t_2|$ and $|\beta_1| \gg |\beta_2|$

(c) Small $|\Delta t_1 - \Delta t_2|$ and $|\beta_1| \approx |\beta_2|$

(d) Large $|\Delta t_1 - \Delta t_2|$ and $|\beta_1| \approx |\beta_2|$

Figure 1.1 shows four plots of normalized[1] $|X(f)|$ (dotted black line) and normalized $|Y(f)|$ (blue line) in [dB]. Match the four graphs (i-iv) to the four cases (a-d).

Problem 2. Consider four vectors

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}, \boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \text{ and } \boldsymbol{v}^{(4)} = \begin{pmatrix} -1 \\ -1 \\ -j \end{pmatrix}.$$

[1] The function is normalized so that the maximum point is ~~1~~ 0 dB.

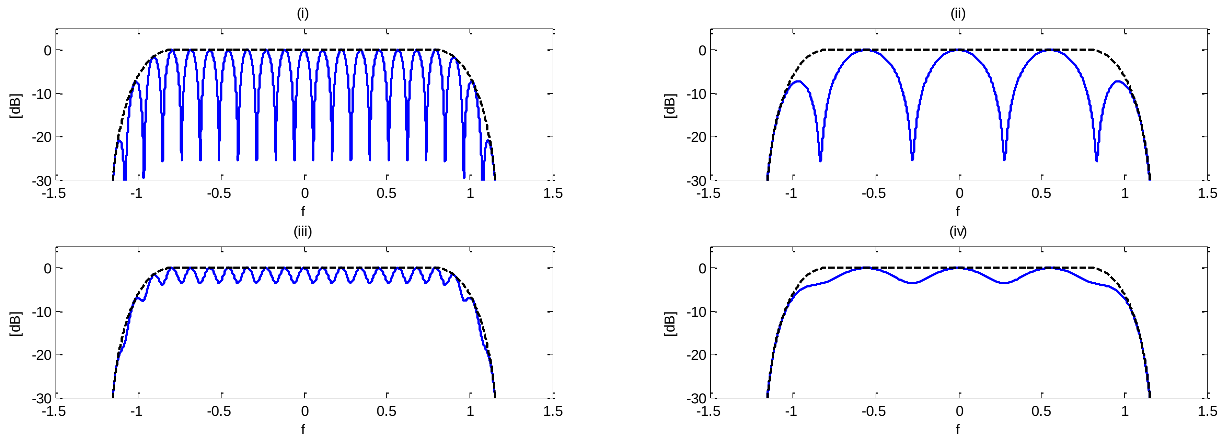


Figure 1.1: Frequency selectivity in the receive spectra (blue line) for two-path channels.

(a) Use the Gram-Schmidt orthogonalization procedure (GSOP) (where the vectors are applied in the order given) to find the orthonormal vectors $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, ... that can be used to represent $\boldsymbol{v}^{(1)}$, $\boldsymbol{v}^{(2)}$, $\boldsymbol{v}^{(3)}$, and $\boldsymbol{v}^{(4)}$.

(b) Use the orthonormal vectors from the previous part to construct the matrix $E = \left[\boldsymbol{e}^{(1)}\ \boldsymbol{e}^{(2)}\ \cdots\right]$. Find the matri~~c~~x C such that $V = EC$ where $V = \left[\boldsymbol{v}^{(1)}\ \boldsymbol{v}^{(2)}\ \boldsymbol{v}^{(3)}\ \boldsymbol{v}^{(4)}\right]$.

Hint: Implement GSOP in MATLAB.

Problem 3. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 1.2. Note that V and T_b are some positive constants. Your answers should be given in terms of them.

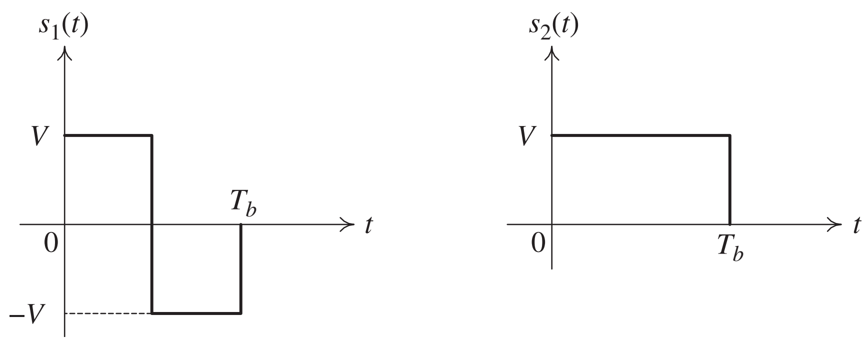


Figure 1.2: Signal set for Question 3

(a) Find the energy in each signal.

(b) Use the Gram-Schmidt orthogonalization procedure (GSOP) (where the signals are applied in the order given) to find two orthonormal functions $\phi_1(t)$ and $\phi_2(t)$ that can be used to represent $s_1(t)$ and $s_2(t)$.

(c) Find the two vectors that represent the two waveforms $s_1(t)$ and $s_2(t)$ in the new (signal) space based on the orthonormal basis found in the previous part. Draw the corresponding constellation.

Problem 4. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 1.3. Note that V, α and T_b are some positive constants.

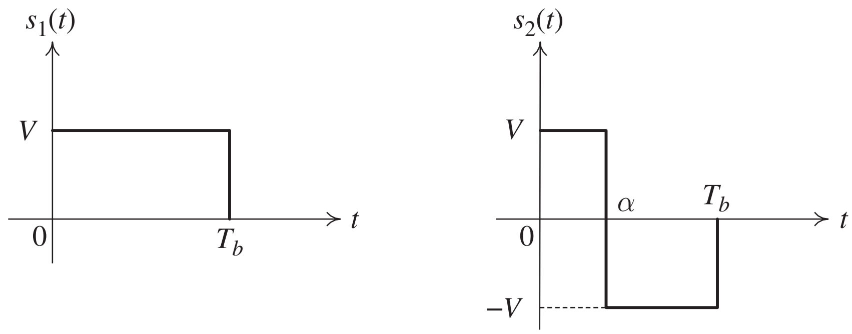


Figure 1.3: Signal set for Question 4

(a) Find the energy in each signal.

(b) Use the Gram-Schmidt orthogonalization procedure (GSOP) (where the signals are applied in the order given) to find two orthonormal functions $\phi_1(t)$ and $\phi_2(t)$ that can be used to represent $s_1(t)$ and $s_2(t)$.

(c) Plot $\phi_1(t)$ and $\phi_2(t)$ when $\alpha = \frac{T_b}{4}$.

(d) Find the two vectors $\boldsymbol{s}^{(1)}$ and $\boldsymbol{s}^{(2)}$ that represent the two waveforms $s_1(t)$ and $s_2(t)$ in the new (signal) space based on the orthonormal basis found in the previous part.

(e) Draw the corresponding constellation when $\alpha = \frac{T_b}{4}$.

(f) Draw $\boldsymbol{s}^{(2)}$ when $\alpha = \frac{k}{10}T_b$ where $k = 1, 2, \dots, 9$.

HW1 Q1 Two-Path Channels

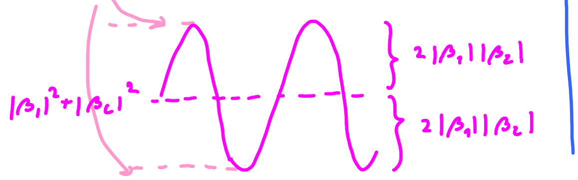
Monday, July 08, 2013 1:53 PM

In class, we have seen that

$$|H(f)|^2 = |\beta_1|^2 + |\beta_2|^2 + 2|\beta_1||\beta_2|\cos\left(2\pi\underbrace{(\Delta t_2 - \Delta t_1)}f + (\angle\beta_1 - \angle\beta_2)\right)$$

$$\max = |\beta_1|^2 + |\beta_2|^2 + 2|\beta_1||\beta_2|$$

$$\min = |\beta_1|^2 + |\beta_2|^2 - 2|\beta_1||\beta_2|$$



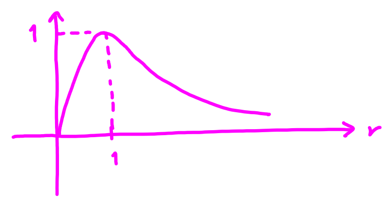
In frequency domain, the "frequency" of the oscillation is determined by $\Delta t_2 - \Delta t_1$.

Large $\Delta t_2 - \Delta t_1$ → more oscillation → (i), (iii)

Small $\Delta t_2 - \Delta t_1$ → less oscillation → (ii), (iv)

$$\text{Depth of fluctuation} = \frac{4|\beta_1||\beta_2|}{|\beta_1|^2+|\beta_2|^2+2|\beta_1||\beta_2|} = \frac{4}{\frac{|\beta_1|}{|\beta_2|}+\frac{|\beta_2|}{|\beta_1|}+2} = \frac{4}{r+\frac{1}{r}+2}$$

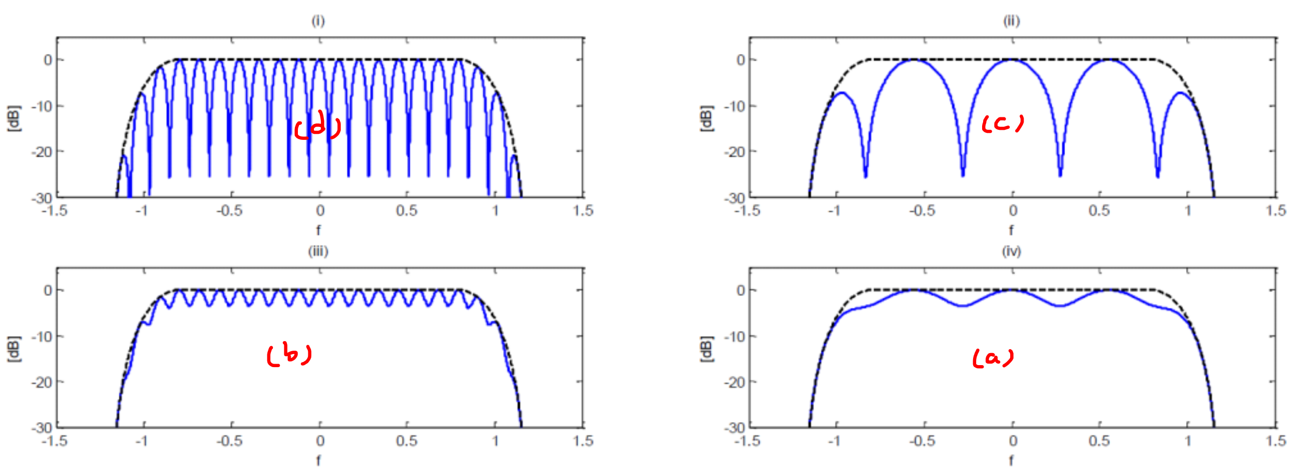
$$r = \left|\frac{\beta_1}{\beta_2}\right|$$



So, depth is large when $|\beta_1| \approx |\beta_2|$ → (i), (ii)

small when $|\beta_1| \gg |\beta_2|$ or $|\beta_1| \ll |\beta_2|$ → (iii), (iv)

Large $\Delta t_2 - \Delta t_1$ Small $\Delta t_2 - \Delta t_1$



$|\beta_1| \approx |\beta_2|$

$|\beta_1| \gg |\beta_2|$ or $|\beta_2| \gg |\beta_1|$

HW1 Q2 GSOP for Complex-Valued Vectors

Monday, July 08, 2013 1:48 PM

From MATLAB

(a)

$$\vec{e}^{(1)} = \frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}, \quad \vec{e}^{(2)} = \frac{1}{2}\begin{pmatrix} 1+j \\ -1+j \\ 0 \end{pmatrix}, \quad \vec{e}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

(b)

$$c = \begin{bmatrix} 2 & -j & 1 & -1 \\ 0 & 1 & -j & j \\ 0 & 0 & 1 & j \end{bmatrix}$$

Detailed Solution

(a)

$$\vec{u}^{(1)} = \vec{v}^{(1)} = \begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}. \qquad \|\vec{u}^{(1)}\|^2 = (1^2+1^2) + (1^2+(-1)^2) + 0 = 2+2 = 4$$

$$\|\vec{u}^{(1)}\| = 2$$

$$\vec{e}^{(1)} = \frac{\vec{u}^{(1)}}{\|\vec{u}^{(1)}\|} = \frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}$$

$$\vec{u}^{(2)} = \vec{v}^{(2)} - \underbrace{\langle \vec{v}^{(2)}, \vec{e}^{(1)} \rangle}\,\vec{e}^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - (-j)\,\frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} j-1 \\ j+1 \\ 0 \end{pmatrix}$$

$$\left(\frac{1}{2}\right)^* \left((1-j) - (1+j)\right)$$

Note that the j is conjugated.

$$= \frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}(-2j) = -j$$

$$\|\vec{u}^{(2)}\|^2 = \frac{1}{2^2}\left((1^2+1^2) + (1^2+(-1)^2) + 0\right)$$

$$\vec{e}^{(2)} = \frac{\vec{u}^{(2)}}{\|\vec{u}^{(2)}\|} = \vec{u}^{(2)} = \frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix} \qquad = \frac{1}{2^2}(4) = 1$$

$$\vec{u}^{(3)} = \underbrace{\vec{v}^{(3)}}_{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} - \underbrace{\langle \vec{v}^{(3)}, \vec{e}^{(1)} \rangle}_{\left(\frac{1}{2}\right)^*\left((1-j)+(1+j)+0\right) = \frac{1}{2}(2) = 1}\,\underbrace{\vec{e}^{(1)}}_{\frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}} - \underbrace{\langle \vec{v}^{(3)}, \vec{e}^{(2)} \rangle}_{\left(\frac{1}{2}\right)^*\left(-j+1-j-1\right) = \frac{1}{2}(-2j) = -j}\,\underbrace{\vec{e}^{(2)}}_{\frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix} + j\,\frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 2 & -1-j & +\,-1+j \\ 2 & -1+j & +\,-1-j \\ -2 & +0 & +\,0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\|\vec{u}^{(3)}\| = \sqrt{0^2+0^2+(-1)^2} = 1$$

$$\vec{e}^{(3)} = \frac{\vec{u}^{(3)}}{\|\vec{u}^{(3)}\|} = \vec{u}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

$$\vec{u}^{(4)} = \underbrace{\vec{v}^{(4)}}_{\begin{pmatrix} -1 \\ -1 \\ -j \end{pmatrix}} - \underbrace{\langle \vec{v}^{(4)}, \vec{e}^{(1)} \rangle}\,\underbrace{\vec{e}^{(1)}}_{\frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}} - \underbrace{\langle \vec{v}^{(4)}, \vec{e}^{(2)} \rangle}\,\underbrace{\vec{e}^{(2)}}_{\frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix}} - \underbrace{\langle \vec{v}^{(4)}, \vec{e}^{(3)} \rangle}\,\underbrace{\vec{e}^{(3)}}_{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}$$

$$\vec{u}^{(4)} = \vec{v}^{(4)} - \underbrace{\langle \vec{v}^{(4)}, \vec{e}^{(1)} \rangle}_{} \vec{e}^{(1)} - \underbrace{\langle \vec{v}^{(4)}, \vec{e}^{(2)} \rangle}_{} \vec{e}^{(2)} - \underbrace{\langle \vec{v}^{(4)}, \vec{e}^{(3)} \rangle}_{} \vec{e}^{(3)}$$

$$= \left(\frac{1}{2}\right)^* \Big(-(1-j) - (1+j)\Big) \qquad = \left(\frac{1}{2}\right)^* \Big(-(-j+1) - (-j-1)\Big) \qquad = (-j)(-1) = j$$

$$= \frac{1}{2}\Big(-1+j-1-j\Big) \qquad = \frac{1}{2}\Big(j-1+j+1\Big)$$

$$= \frac{-2}{2} = -1 \qquad = \frac{1}{2}\Big(2j\Big) = j$$

$$= \frac{1}{2}\begin{pmatrix} -2 & +1+j & -(-1+j) & +0 \\ -2 & +1-j & -(-1-j) & +0 \\ -2j & +0 & -\ 0 & +2j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leftarrow \text{so, no } \vec{u}^{(4)} \text{ and hence no } \vec{e}^{(4)}.$$

(b) From the procedure above, we get

$$\vec{u}^{(1)} = \vec{v}^{(1)} \quad \Rightarrow \quad \vec{v}^{(1)} = \vec{u}^{(1)} = \|\vec{u}^{(1)}\| \vec{e}^{(1)} = 2\vec{e}^{(1)} = E\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{u}^{(2)} = \vec{v}^{(2)} - (-j)\vec{e}^{(1)} \quad \Rightarrow \quad \vec{v}^{(2)} = (-j)\vec{e}^{(1)} + \|\vec{u}^{(2)}\|\vec{e}^{(2)} = (-j)\vec{e}^{(1)} + 1\,\vec{e}^{(2)} = E\begin{pmatrix} -j \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{u}^{(3)} = \vec{v}^{(3)} - 1\,\vec{e}^{(1)} - (-j)\vec{e}^{(2)} \quad \Rightarrow \quad \vec{v}^{(3)} = (1)\vec{e}^{(1)} + (-j)\vec{e}^{(2)} + \|\vec{u}^{(3)}\|\vec{e}^{(3)}$$

$$= (1)\vec{e}^{(1)} + (-j)\vec{e}^{(2)} + (1)\vec{e}^{(3)} = E\begin{pmatrix} 1 \\ -j \\ 1 \end{pmatrix}$$

$$\underbrace{\vec{u}^{(4)}}_{0} = \vec{v}^{(4)} - (-1)\vec{e}^{(1)} - (j)\vec{e}^{(2)} - (j)\vec{e}^{(3)} \quad \Rightarrow \quad \vec{v}^{(4)} = (-1)\vec{e}^{(1)} + (j)\vec{e}^{(2)} + (j)\vec{e}^{(3)} = E\begin{pmatrix} -1 \\ j \\ j \end{pmatrix}$$

$$\text{so,} \quad V = \left[\vec{v}^{(1)} \ \vec{v}^{(2)} \ \vec{v}^{(3)} \ \vec{v}^{(4)}\right] = E \underbrace{\boxed{\begin{bmatrix} 2 & -j & 1 & -1 \\ 0 & 1 & -j & j \\ 0 & 0 & 1 & j \end{bmatrix}}}_{C}.$$

HW1 Q3 Signal Space and Constellation

Monday, July 08, 2013 10:39 AM

(a)

$$\mathcal{E}_1 = \mathcal{E}_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2\, dt = \int_0^{T_b} V^2\, dt = V^2 T_b.$$

$$\mathcal{E}_2 = \mathcal{E}_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2\, dt = \int_0^{T_b} V^2\, dt = V^2 T_b.$$

$$\equiv \mathcal{E}$$

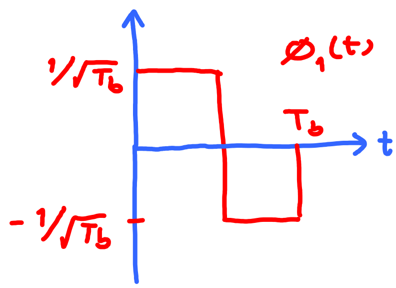
(b)

$$u_1(t) = s_1(t).$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} = \frac{1}{\sqrt{\mathcal{E}}}\, s_1(t)$$

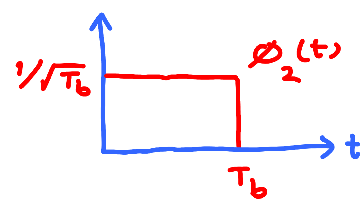
$$\frac{V}{\sqrt{\mathcal{E}}} = \frac{V}{V\sqrt{T_b}} = \frac{1}{\sqrt{T_b}}$$

$$= \begin{cases} 1/\sqrt{T_b}, & 0 \le t \le \frac{T_b}{2} \\ -1/\sqrt{T_b}, & \frac{T_b}{2} < t < T_b \\ 0, & \text{otherwise} \end{cases}$$



$$u_2(t) = s_2(t) - \mathrm{proj}_{u_1} s_2 = s_2(t) - \frac{\overbrace{\langle s_2, s_1\rangle}^{0}}{\langle s_1, s_1\rangle} s_1(t) = s_2(t)$$

$$\phi_2(t) = \frac{u_2(t)}{\sqrt{\mathcal{E}_{u_2}}} \underset{u_2 = s_2}{=} \frac{s_2(t)}{\sqrt{\mathcal{E}_2}} = \frac{1}{\sqrt{\mathcal{E}}}\, s_2(t)$$

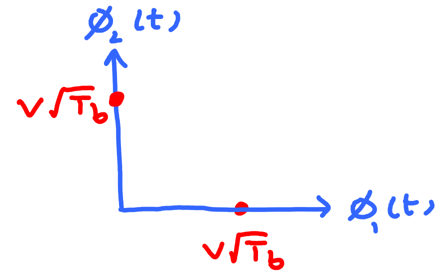


$$= \begin{cases} 1/\sqrt{T_b}, & 0 \le t \le T_b, \\ 0, & \text{otherwise.} \end{cases}$$

(c)

$$s_1(t) = \sqrt{\mathcal{E}}\, \phi_1(t) \Rightarrow \vec{s}^{(1)} = \begin{pmatrix} \sqrt{\mathcal{E}} \\ 0 \end{pmatrix} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix}.$$

$$s_2(t) = \sqrt{\mathcal{E}}\, \phi_2(t) \Rightarrow \vec{s}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{\mathcal{E}} \end{pmatrix} = \begin{pmatrix} 0 \\ V\sqrt{T_b} \end{pmatrix}.$$



HW1 Q4 Signal Space and Constellation

Monday, July 08, 2013 11:17 AM

(a)

$$\mathcal{E}_1 = \mathcal{E}_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

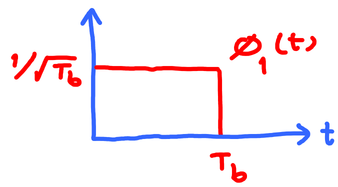
$$\mathcal{E}_2 = \mathcal{E}_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$$\equiv \mathcal{E}$$

(b)

$u_1(t) = s_1(t).$

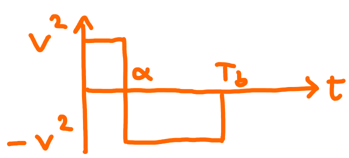
$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} = \frac{1}{\sqrt{\mathcal{E}}} s_1(t) = \begin{cases} 1/\sqrt{T_b}, & 0 < t < T_b \\ 0, & \text{otherwise.} \end{cases}$$



$$\frac{V}{\sqrt{\mathcal{E}}} = \frac{V}{V\sqrt{T_b}} = \frac{1}{\sqrt{T_b}}$$

$$u_2(t) = s_2(t) - \mathrm{proj}_{u_1} s_2 = s_2(t) - \frac{\langle s_2, s_1 \rangle}{\langle s_1, s_1 \rangle} s_1(t)$$

$$\langle s_2, s_1 \rangle = \int \quad dt$$



$$= \alpha V^2 - (T_b - \alpha) V^2 = 2\alpha V^2 - T_b V^2$$

$$u_2(t) = s_2(t) - \frac{2\alpha \cancel{V^2} - T_b \cancel{V^2}}{\cancel{V^2} T_b} s_1(t) = s_2(t) - \left(\frac{2\alpha}{T_b} - 1\right) s_1(t)$$

$$= s_2(t) + \underbrace{\left(1 - \frac{2\alpha}{T_b}\right)} s_1(t)$$

When $\alpha = \frac{T_b}{2}$, this term $= 0$ and $u_2(t) = s_2(t)$.

When $\alpha < \frac{T_b}{2}$, this term is > 0. So, $s_2(t)$ will be shifted up.

When $\alpha > \frac{T_b}{2}$, this term is < 0. So, $s_2(t)$ will be shifted down.

$$\begin{cases} V + \left(1 - \frac{2\alpha}{T_b}\right) V & \text{for } 0 < t < \alpha, \end{cases}$$

$$= \begin{cases} V + \left(1-\frac{2\alpha}{T_b}\right)V & \text{for } 0<t<\alpha, \\ -V + \left(1-\frac{2\alpha}{T_b}\right)V & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left(2-\frac{2\alpha}{T_b}\right)V & \text{for } 0<t<\alpha, \\ -\frac{2\alpha}{T_b}V & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases} = 2V \times \begin{cases} 1-\frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

Note: The shift amount is just enough to make $\int u_2 = 0$

$$\left[\left(1-\frac{\alpha}{T_b}\right)\alpha + \left(-\frac{\alpha}{T_b}\right)(T_b-\alpha) = \alpha - \frac{\alpha^2}{T_b} - \alpha + \frac{\alpha^2}{T_b} = 0.\right]$$

$$\mathcal{E}_{u_2} = 4V^2\left(\underbrace{\left(1-\frac{\alpha}{T_b}\right)^2}_{1-\frac{2\alpha}{T_b}+\frac{\alpha^2}{T_b^2}}\alpha + \left(-\frac{\alpha}{T_b}\right)^2(T_b-\alpha)\right) = 4V^2\left(\alpha - \cancel{\frac{2\alpha^2}{T_b}} + \cancel{\frac{\alpha^3}{T_b^2}} + \cancel{\frac{\alpha^2}{T_b}} - \cancel{\frac{\alpha^3}{T_b^2}}\right)$$

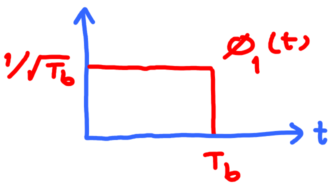
$$= 4V^2\alpha\left(1-\frac{\alpha}{T_b}\right) \qquad \Rightarrow \sqrt{\mathcal{E}_{u_2}} = 2V\sqrt{\alpha\left(1-\frac{\alpha}{T_b}\right)}$$

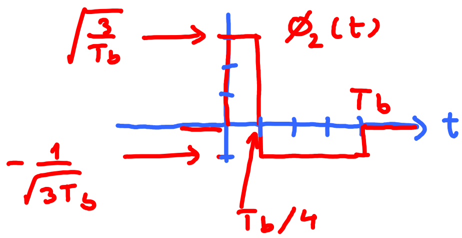
$$\phi_2(t) = \frac{u_2(t)}{\sqrt{\mathcal{E}_{u_2}}} = \frac{2\cancel{V}}{\cancel{2V}\sqrt{\alpha\left(1-\frac{\alpha}{T_b}\right)}} \begin{cases} 1-\frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \frac{1}{\sqrt{\alpha\left(1-\frac{\alpha}{T_b}\right)}} \times \begin{cases} 1-\frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

(c) When $\alpha = \frac{1}{4}T_b$, $\quad = \frac{1}{\sqrt{\frac{T_b}{4}\left(1-\frac{1}{4}\right)}} = \frac{4}{\sqrt{3T_b}}$

$$\phi_2(t) = \frac{4}{\sqrt{3T_b}} \times \begin{cases} 3/4 & \text{for } 0<t<\frac{T_b}{4} \\ -1/4 & \text{for } \frac{T_b}{4}<t<T_b \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \sqrt{3/T_b}, & \text{for } 0<t<\frac{T_b}{4}, \\ -\frac{1}{\sqrt{3T_b}}, & \text{for } \frac{T_b}{4}<t<T_b, \\ 0, & \text{otherwise.} \end{cases}$$





(d)

$$s_1(t) = \sqrt{\mathcal{E}_1}\,\phi_1(t) = \sqrt{\mathcal{E}}\,\phi_1(t) = V\sqrt{T_b}\,\phi_1(t) \Rightarrow \vec{s}^{(1)} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\mathcal{E}} \\ 0 \end{pmatrix}.$$

$$s_2(t) = u_2(t) - \left(1 - 2\frac{\alpha}{T_b}\right) s_1 = \sqrt{\mathcal{E}_{u_2}}\,\phi_2(t) - \left(1 - 2\frac{\alpha}{T_b}\right)\sqrt{\mathcal{E}}\,\phi_1(t)$$

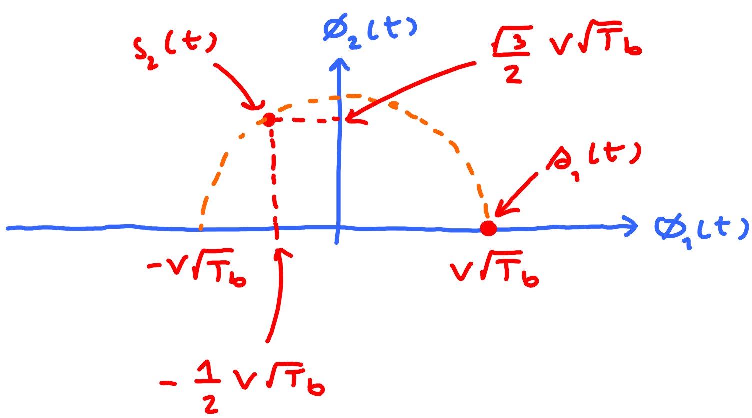
$$= 2V\sqrt{\alpha\left(1 - \frac{\alpha}{T_b}\right)}\,\phi_2(t) - \left(1 - 2\frac{\alpha}{T_b}\right) V\sqrt{T_b}\,\phi_1(t).$$

$$\Rightarrow \vec{s}^{(2)} = \begin{pmatrix} -\left(1 - 2\frac{\alpha}{T_b}\right) \\ 2\sqrt{\frac{\alpha}{T_b}\left(1 - \frac{\alpha}{T_b}\right)} \end{pmatrix} \times V\sqrt{T_b} = \sqrt{\mathcal{E}} \begin{pmatrix} 2r - 1 \\ 2\sqrt{r(1-r)} \end{pmatrix} \quad \text{where } r = \frac{\alpha}{T_b}.$$

(e)

When $\alpha = \frac{T_b}{4}$, $s_2(t) = 2V\sqrt{\frac{T_b}{4}\left(\frac{3}{4}\right)}\,\phi_2(t) - \left(1 - \frac{1}{2}\right) V\sqrt{T_b}\,\phi_1(t)$

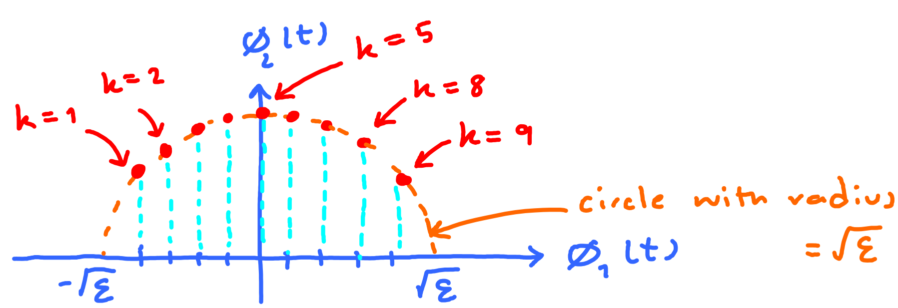
$$= \frac{\sqrt{3}}{2} V\sqrt{T_b}\,\phi_2(t) - \frac{1}{2} V\sqrt{T_b}\,\phi_1(t)$$



(f)

Note that $(2r-1)^2 + \left(2\sqrt{r(1-r)}\right)^2 = 4r^2 - 4r + 1 + 4r - 4r^2 = 1.$

$$r = \frac{\alpha}{T_b} = \frac{k}{10}$$



ECS 452: Digital Communication Systems	**2013/1**

HW 2 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{3, -3\}$ with $p_1 = P[S = -3] = 0.3$ and $p_2 = P[S = 3] = 0.7$. The message is corrupted by an independent additive exponential noise N whose pdf is

$$f_N(n) = \begin{cases} \frac{1}{2}e^{-n/2}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the MAP detector $\hat{s}_{\mathrm{MAP}}(r)$.

(b) Indicate the decision regions of the MAP detector in part (a).

(c) Consider a detector of the form

$$\hat{s}(r) = \begin{cases} 3, & r > \tau, \\ -3, & r \leq \tau \end{cases}$$

for some threshold τ. Find and then plot the probability of (symbol detection) error for this detector as a function of τ. Hint: The plots from actual simulation are shown in class. The same plots are shown in Figure 2.1.

(d) Evaluate the error probability of the MAP detector.

(e) Evaluate the error probability of the ML detector.

Problem 2. Repeat parts (a)-(d) of Question 1 but now the noise is uniform on $[-4, 4]$.

Problem 3. In a ternary signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-1, 1, 4\}$ with $p_1 = P[S = -1] = 0.3 = p_2 = P[S = 1]$ and $p_3 = P[S = 4] = 0.4$. The message is corrupted by an independent additive Gaussian noise $N \sim \mathcal{N}(0, 2)$.

(a) Find the average signal energy[1] E_s.

Note that

$$E_s = \sum_i p_i |s_i|^2.$$

[1] Same as "average symbol energy" or "average energy per symbol" or "average energy per signal"

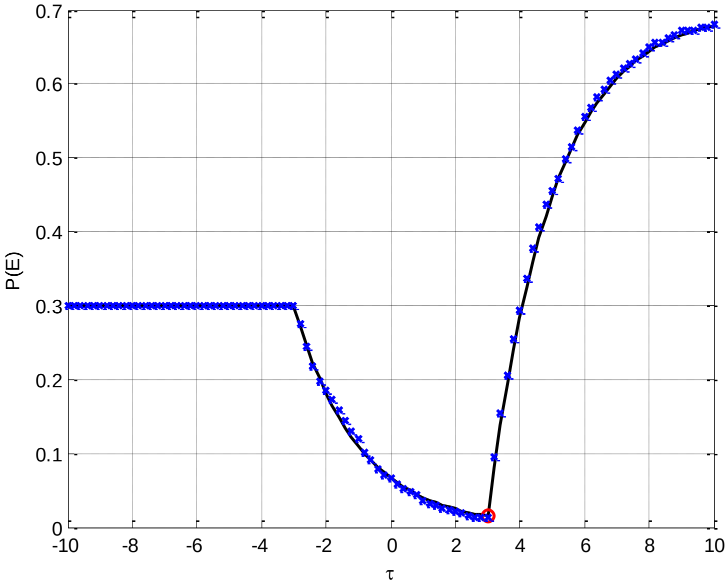


Figure 2.1: $P(\mathcal{E})$ for Exponential Noise in Question 1

(b) Find the MAP detector $\hat{s}_{\mathrm{MAP}}(r)$.

(c) Indicate the decision regions of the MAP detector in part (b).

(d) Evaluate the error probability of the MAP detector.

Problem 4. In a ternary signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-1, 1, 4\}$ with $p_1 = P[S = -1] = 0.41$, $p_2 = P[S = 1] = 0.08$ and $p_3 = P[S = 4] = 0.51$. The message is corrupted by an independent additive Gaussian noise $N \sim \mathcal{N}(0, 2)$.

(a) Find the average signal energy E_s.

(b) If the MAP detector is used, find $P(\mathcal{E}|S = 1)$; that is, find the probability of (decoding) error given that $S = 1$ was transmitted.

Problem 5. In a **standard** quaternary signaling scheme, the message S is equiprobably selected from the alphabet set $\mathcal{S} = \left\{-\frac{3d}{2}, -\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}\right\}$. The message is corrupted by an independent additive exponential noise N whose pdf is

$$f_N(n) = \begin{cases} \lambda e^{-\lambda n}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the average symbol energy.

(b) Find the average energy per bit.

(c) Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

(d) Evaluate the error probability of the MAP detector.

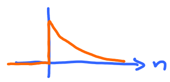
(e) Let $\lambda = \frac{1}{\sigma}$. (This is to set $\operatorname{Var} N = \sigma^2$ as in the case for Gaussian noise.) Plot $\frac{E_b}{\sigma^2}$ vs. probability of error $P(\mathcal{E})$. Consider $\frac{E_b}{\sigma^2}$ from -30 to 10 dB.

HW2 Q1: 1-D MAP Detector and Exponential Noise

Monday, July 15, 2013 10:58 AM

Exponential Noise

$$f_N(n) = \begin{cases} \lambda e^{-\lambda n}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$



(Note that $\mathbb{E}N = \frac{1}{\lambda}$ and $\operatorname{Var} N = \frac{1}{\lambda^2}$)

MATLAB use $\mathbb{E}N$ as the parameter instead of λ

$$P[N > n] = \begin{cases} e^{-\lambda n}, & n \geq 0, \\ \cancel{0}\, 1, & \text{otherwise.} \end{cases}$$

In this question, $\lambda = \frac{1}{2}$ and $\mathbb{E}N = \frac{1}{\lambda} = 2$.

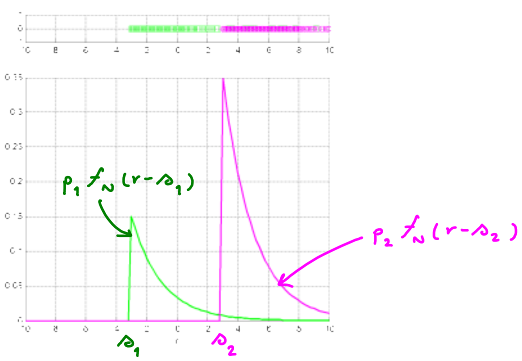
(a)

MAP Detector : Recall that $\hat{s}_{MAP}(r) = \arg\max_{s} p_s f_N(r-s)$

This is true regardless of the pdf of the noise

Here, there are two possible values for s.

So we compare $\underset{0.3}{\underset{\uparrow}{p_1}} f_N(r - \underset{-3}{\underset{\uparrow}{s_1}})$ and $\underset{0.7}{\underset{\uparrow}{p_2}} f_N(r - \underset{3}{\underset{\uparrow}{s_2}})$



From the graph, it is clear that

1. $R < s_1$ is impossible. So, the detector can do anything in this region without affecting its performance.

2. When $s_1 < r < s_2$, $p_1 f_N(r-s_1) > p_2 f_N(r-s_2)$.
So, in this region, $\hat{s}_{MAP}(r) = s_1$.

3. When $r > s_2$, $p_1 f_N(r-s_1) < p_2 f_N(r-s_2)$.
So, in this region, $\hat{s}_{MAP}(r) = s_2$.

Conclusion:

$$\hat{s}_{MAP}(r) = \begin{cases} s_1, & s_1 < r < s_2, \\ s_2, & r > s_2, \\ \text{anything}, & \text{otherwise} \end{cases} \underset{\text{Simplification}}{\overset{\uparrow}{=}} \begin{cases} s_1, & r < s_2 \\ s_2, & r \geq s_2 \end{cases}$$

$\tau^* = s_2$

$$= \begin{cases} -3, & r < 3 \\ 3, & r \geq 3 \end{cases}$$

(b) $D_i = \{ r : \hat{s}_{MAP}(r) = s_i \} \Rightarrow \begin{cases} D_1 = (-\infty, 3) \\ D_2 = [3, \infty) \end{cases}$

(c) Probability of error:

In class, we show that for the detector of the form

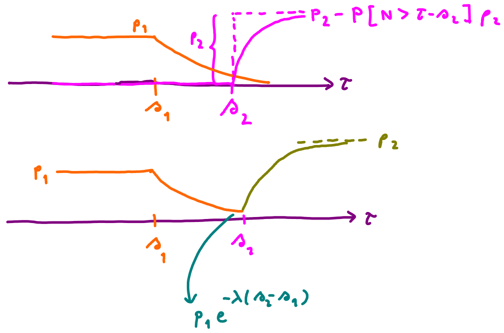
$$\hat{s}(r) = \begin{cases} s_1, & r < \tau, \\ s_2, & r \geq \tau, \end{cases}$$

$$P(\mathcal{E}) = p_1 P[N \geq \tau - s_1] + p_2 P[N < \tau - s_2]$$

$$= p_1 P[N \geq \tau - s_1] + p_2 (1 - P[N \geq \tau - s_2])$$

For exponential noise, $P[N > n] = \int_n^\infty f_N(n)\,dn = \begin{cases} e^{-\lambda n}, & n \geq 0 \\ 1, & n < 0 \end{cases}$

Therefore,



$$P(\mathcal{E}) = \begin{cases} p_1, & \tau < s_1, \\ p_1 e^{-\lambda(\tau - s_1)}, & s_1 \leq \tau \leq s_2, \\ p_1 e^{-\lambda(\tau - s_1)} + p_2\left(1 - e^{-\lambda(\tau - s_2)}\right) & \tau > s_2, \end{cases}$$

$$= \begin{cases} 0.3, & \tau < -3, \\ 0.3\, e^{-\frac{1}{2}(\tau+3)} & -3 \leq \tau \leq 3, \\ 0.3\, e^{-\frac{1}{2}(\tau+3)} - 0.7\, e^{-\frac{1}{2}(\tau-3)} + 0.7 & \tau > 3 \end{cases}$$

(d) Method (i), plug-in $\tau = s_2$ into the expression derived in part (c)

$$\Rightarrow P(\mathcal{E}) = p_1 e^{-\lambda(s_2 - s_1)} = 0.3\, e^{-\frac{1}{2}(3-(-3))} = 0.3\, e^{-3} \approx 0.0149$$

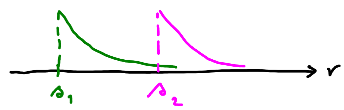
Method (ii), $P(\mathcal{E}) = \underbrace{P(\mathcal{E} \mid S = s_1)}_{= P[N > s_2 - s_1] = p_1 e^{-\lambda(s_2 - s_1)}} p_1 + \underbrace{P(\mathcal{E} \mid S = s_2)}_{=0} p_2$

$=0$ because N is always positive and the MAP detector always detect s_2 when $R > s_2$

(e) Recall that ML detector ignores the prior probabilities; that is

$$\hat{s}_{ML}(r) = \arg\max_{s} f_N(r - s).$$

So, we compare $f_N(r - s_1)$ and $f_N(r - s_2)$



Observe that all the discussion we have in part (a) still works here.

Therefore,

$$\hat{s}_{ML}(r) = \hat{s}_{MAP}(r) = \begin{cases} -3, & r < 3 \\ 3, & r \geq 3 \end{cases}$$

HW2 Q2: 1-D MAP Detector and Uniform Noise

Monday, July 15, 2013 1:33 PM

Recall that the uniform pdf on $[a,b]$ is given by

$$f_N(n) = \begin{cases} \frac{1}{b-a}, & a < n < b, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $a = 4$ and $b = -4$. So,

$$f_N(n) = \begin{cases} 1/8, & -4 < n < 4, \\ 0, & \text{otherwise.} \end{cases}$$

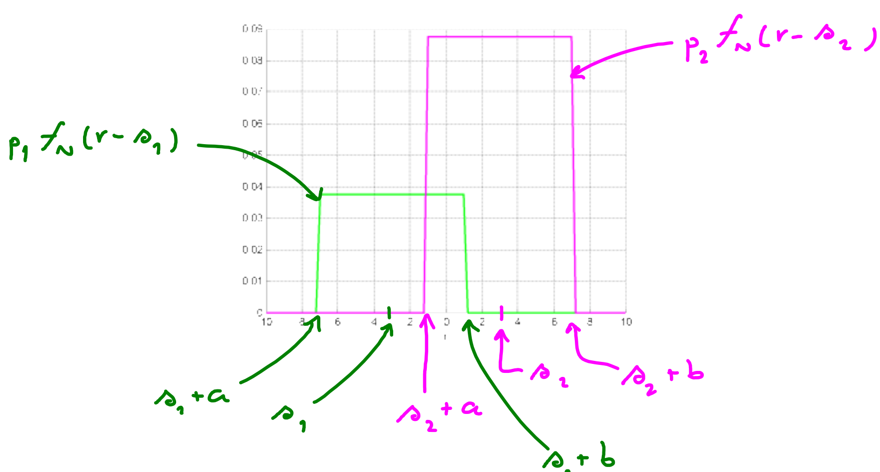
(a) Again, the MAP detector is given by

$$\hat{s}_{MAP}(r) = \arg\max_{s} p_s f_N(r - s).$$

This is true regardless of the pdf of the noise

Here, there are two possible values for s.

So we compare $\underset{\uparrow \, 0.3}{p_1} f_N(r - \underset{\uparrow \, -3}{s_1})$ and $\underset{\uparrow \, 0.7}{p_2} f_N(r - \underset{\uparrow \, 3}{s_2})$



Observe that

① when $s_2 + a < r < s_2 + b$, $p_1 f_N(r - s_1) < p_2 f_N(r - s_2)$.

Therefore, $\hat{s}_{MAP}(r) = s_2$ in this region.

② when $s_1 + a < r < s_2 + a$, $p_1 f_N(r - s_1) > p_2 f_N(r - s_2)$.

Therefore, $\hat{s}_{MAP}(r) = s_1$ in this region.

③ when $r < s_1 + a$ or $r > s_2 + b$, the pdf in both cases are 0.

So, these are the impossible regions. The received signal R won't fall in these regions.

Therefore, it does not matter how the detector behaves in this region.

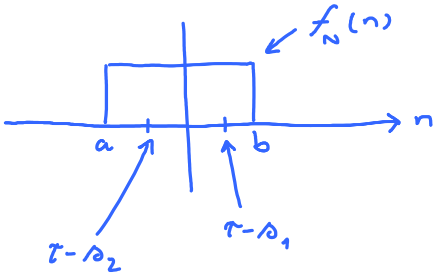
Conclusion: $\hat{s}_{MAP}(r) = \begin{cases} s_1, & s_1 + a < r < s_2 + a, \\ s_2, & s_2 + a < r < s_2 + b, \\ \text{anything}, & \text{otherwise} \end{cases}$

simplification $\searrow$ $= \begin{cases} s_1, & r < s_2 + a \\ s_2, & r \geq \underbrace{s_2 + a}_{\tau^* = s_2 + a} \end{cases} = \begin{cases} -3, & r < -1 \\ 3, & r \geq -1 \end{cases}$

(b) $D_1 = \{ r : \hat{s}(r) = s_1 \} = (-\infty, -1)$

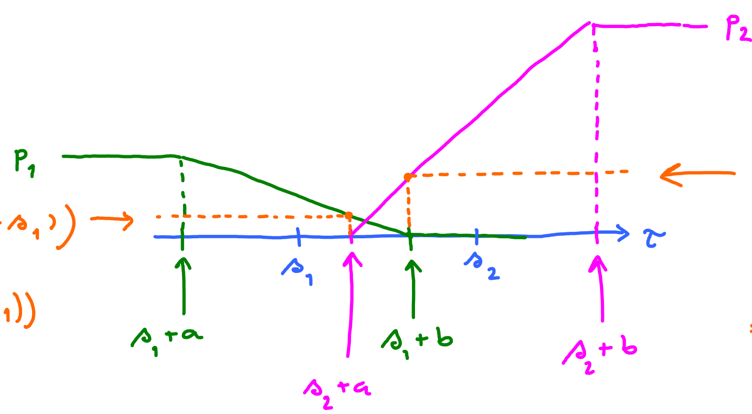
$D_2 = \{ r : \hat{s}(r) = s_2 \} = [1, \infty)$

(c) Recall that $P(\mathcal{E}) = p_1 P[N \geq \tau - s_1] + p_2 P[N < \tau - s_2]$



$P[N \geq \tau - s_1] = \begin{cases} 1, & \tau - s_1 \leq a \\ \frac{1}{b-a}(b - (\tau - s_1)) & a \leq \tau - s_1 \leq b \\ 0 & \tau - s_1 \geq b \end{cases}$

$P[N < \tau - s_2] = \begin{cases} 0, & \tau - s_2 \leq a \\ \frac{1}{b-a}(\tau - s_2 - a) & a \leq \tau - s_2 \leq b \\ 1, & \tau - s_2 \geq b \end{cases}$



$p_1 \frac{1}{b-a} (b - (s_2 + a - s_1)) \rightarrow$

$= p_1 \frac{1}{b-a} (b - a - (s_2 - s_1))$

$= p_1 \left(1 - \frac{s_2 - s_1}{b-a} \right)$

$\leftarrow p_2 \frac{1}{b-a} (s_1 + b - s_2 - a)$

$= p_2 \frac{1}{b-a} (b - a - (s_2 - s_1))$

$= p_2 \left(1 - \frac{s_2 - s_1}{b-a} \right)$

Here, $p_1 = 0.3$, $p_2 = 0.7$, $s_2 - s_1 = 3 - (-3) = 6$, $b - a = 4 - (-4) = 8$

$s_1 + a = -3 - 4 = -7$ $\quad$ $s_1 + b = -3 + 4 = 1$

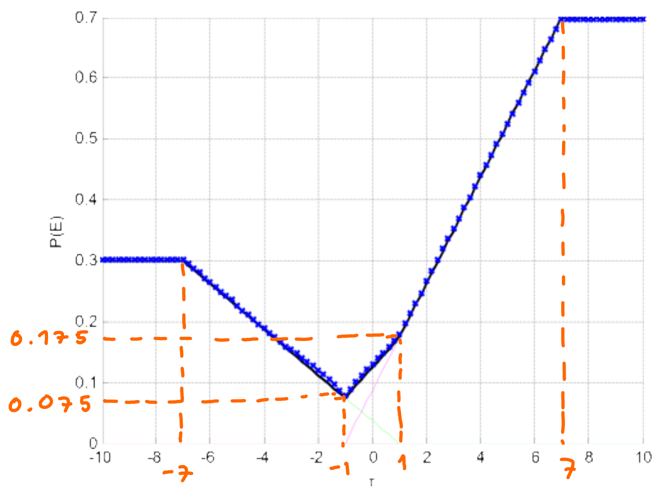
Here, $p_1 = 0.3$, $p_2 = 0.7$, $s_2 - s_1 = 3-(-3) = 6$, $b - a = 4-(-4) = 8$

$$s_1 + a = -3-4 = -7 \qquad s_1 + b = -3+4 = 1$$

$$s_2 + a = 3-4 = -1 \qquad s_2 + b = 3+4 = 7$$

$$p_1\left(1 + \frac{s_2 - s_1}{b-a}\right) = 0.3\left(1 - \frac{3}{4}\right) = \frac{0.3}{4} = 0.075$$

$$p_2\left(1 - \frac{s_2 - s_1}{b-a}\right) = 0.7\left(1 - \frac{3}{4}\right) = \frac{0.7}{4} = 0.175$$



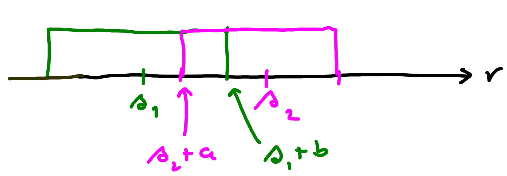
(d) When $\tau = -1$, part (c) tells us that $P(\mathcal{E}) = 0.075$.

τ^* for MAP detector in part (a).

(e) Recall that ML detector ignores the prior probabilities; that is

$$\hat{s}_{ML}(r) = \arg\max_{s} f_N(r - s).$$

So, we compare $f_N(r - s_1)$ and $f_N(r - s_2)$



Observe that when $r < s_2 + a$ or $r > s_1 + b$, we got the same conclusion as in part (a):

$$\hat{s}_{ML}(r) = \begin{cases} s_1, & r < s_2 + a, \\ s_2, & r > s_1 + b. \end{cases}$$

However, when $s_2 + a < r < s_1 + b$, the two graphs are the same.
Therefore, the ML detector is free to choose s_1 or s_2 in this region.

If $p_1 = p_2$, this will not affect the overall $P(\mathcal{E})$. In fact, you will get MAP detector.

HW2 Q3: 1-D Multi-Level MAP Detector and Gaussian Noise

Wednesday, July 24, 2013 2:31 PM

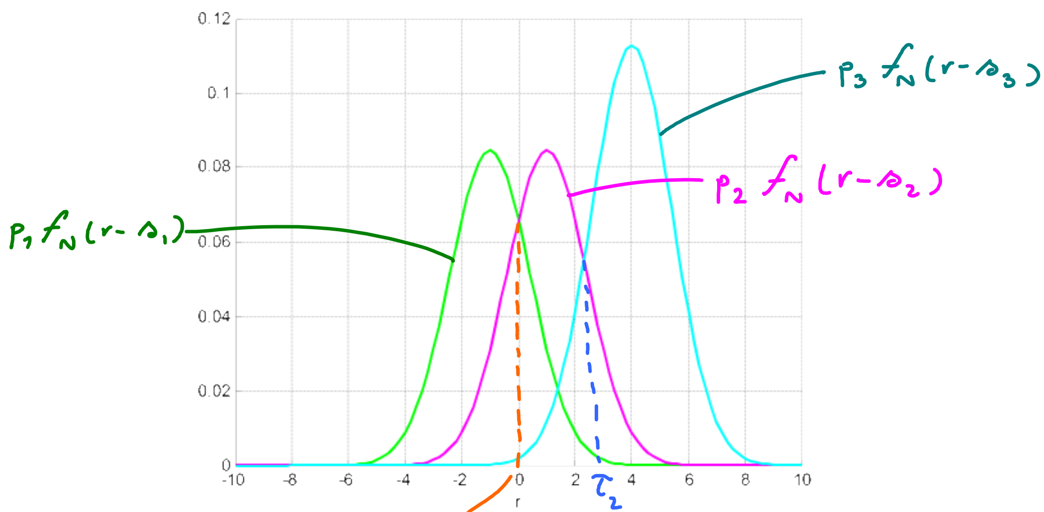
Three-point Constellation

$$S = \{\underset{s_1}{-1}, \underset{s_2}{1}, \underset{s_3}{4}\}$$

$$\underbrace{P[S=-1]}_{p_1} = 0.3 = \underbrace{P[S=1]}_{p_2}$$

$$\underbrace{P[S=4]}_{p_3} = 0.4$$

$$\text{Average energy per symbol} = (-1)^2 \times 0.3 + 1^2 \times 0.3 + 4^2 \times 0.4$$
$$= 0.3 + 0.3 + 6.4 = 7$$



To find τ_1, we find r such that $p_1 f_N(r-s_1) = p_2 f_N(r-s_2)$.

$$p_1 \cancel{\frac{1}{\sqrt{2\pi}\Delta}} e^{-\frac{1}{2}\left(\frac{r-s_1}{\Delta}\right)^2} = p_2 \cancel{\frac{1}{\sqrt{2\pi}\Delta}} e^{-\frac{1}{2}\left(\frac{r-s_2}{\Delta}\right)^2}$$

$$\frac{p_1}{p_2} = e^{-\frac{1}{2\Delta^2}\left((r-s_2)^2 - (r-s_1)^2\right)}$$

$$2\Delta^2 \ln \frac{p_1}{p_2} = (2r - s_2 - s_1)(s_2 - s_1)$$

$$r = \frac{\Delta^2}{s_2 - s_1} \ln \frac{p_1}{p_2} + \frac{s_1 + s_2}{2}$$

$\underbrace{(s_2 - s_1 \quad p_2 \quad 2}$

This is the same formula that we derived in lecture to find τ^* for the MAP detector when $M=2$.

Here, $p_1 = p_2$. So, $\tau_1 = \frac{s_1+s_2}{2} = \frac{-1+1}{2} = 0$

Similarly, we can find τ_2 by $\frac{\Delta^2}{s_3 - s_2} \ln \frac{p_3}{p_2} + \frac{s_2+s_3}{2} = 2.3082$

The MAP detector is given by

$$\hat{s}_{MAP}(r) = \begin{cases} s_1, & r \le \tau_1 \\ s_2, & \tau_1 < r \le \tau_2 \\ s_3, & r > \tau_2 \end{cases}$$

$$= \begin{cases} -1, & r \le 0 \\ 1 & 0 < r \le 2.3082 \\ 4 & r > 2.3082 \end{cases}$$

$$D_1 = \{r : \hat{s}_{MAP}(r) = s_1\} = (-\infty, 0]$$
$$D_2 = \{r : \hat{s}_{MAP}(r) = s_2\} = (0, 2.3082]$$
$$D_3 = \{r : \hat{s}_{MAP}(r) = s_3\} = (2.3082, \infty)$$

$$P(\mathcal{E}) = \underbrace{P(\mathcal{E} \mid S = s_1)}_{= P[R > \tau_1 \mid S = s_1]} p_1 + \underbrace{P(\mathcal{E} \mid S = s_2)}_{P[R \le \tau_1 \mid S = s_2] + P[R > \tau_2 \mid S = s_2]} p_2 + \underbrace{P(\mathcal{E} \mid S = s_3)}_{P[R \le \tau_2 \mid S = s_3]} p_3$$

$$= P[N > \tau_1 - s_1]\, p_1 + \left(P[N \le \tau_1 - s_2] + P[N > \tau_2 - s_2]\right) p_2 + P[N \le \tau_2 - s_3]\, p_3$$

$$= p_1 Q\left(\frac{\tau_1 - s_1}{\Delta}\right) + \left(Q\left(\frac{s_2 - \tau_1}{\Delta}\right) + Q\left(\frac{\tau_2 - s_2}{\Delta}\right)\right) p_2 + p_3 Q\left(\frac{s_3 - \tau_2}{\Delta}\right)$$

$$= 0.2434$$

HW2 Q4: 1-D Multi-Level MAP Detector and Gaussian Noise

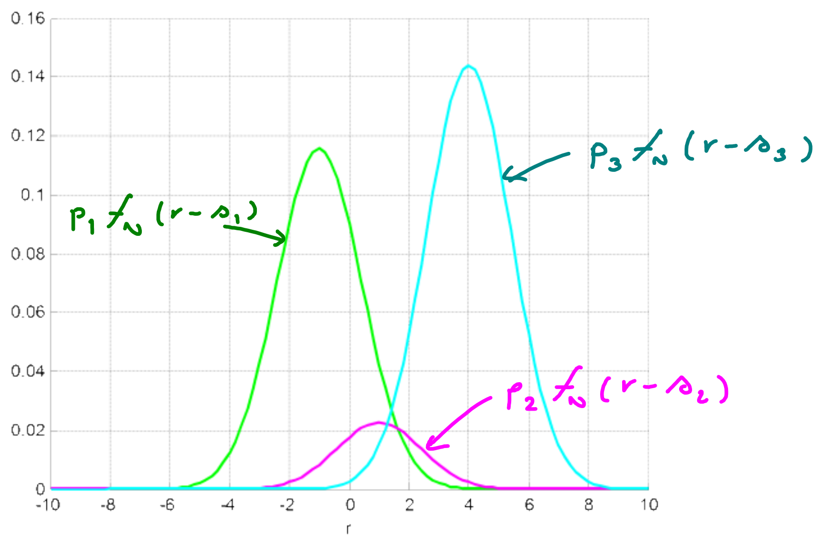
Wednesday, July 24, 2013 3:13 PM

(a) Average symbol energy :

$$E_s = 0.41 \times (-1)^2 + 0.08 \times (1)^2 + 0.51 \times 4^2 = 8.65$$

(b)

Note that $p_2 f_N(r-s_2)$ is always below $p_1 f_N(r-s_1)$ and $p_3 f_N(r-s_3)$.



So, the MAP detector will never choose s_2 as its answer.

Therefore, $P(\mathcal{E} \mid S = s_2) = 1$.

Note that although this may seem bad, it is in fact not so bad. The key idea is to realize that p_2 is very small and hence the contribution to the overall $P(\mathcal{E})$ is also small.

HW2 Q5: 1-D Standard Multi-Level MAP Detector and Expo Noise

Wednesday, July 24, 2013 3:35 PM

(a) Here there are four ($M=4$) possible values for s : $-\frac{3d}{2}, -\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}$

They are equally likely. Therefore $p_i = \frac{1}{4}$.

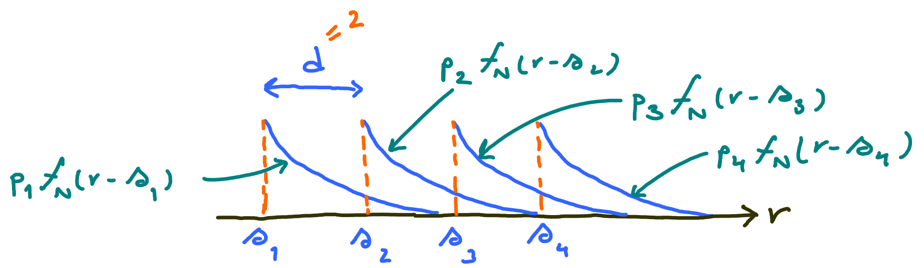
$$E_s = \sum_i p_i |s_i|^2 = \frac{1}{4}\left(\left(-\frac{3}{2}d\right)^2 + \left(-\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 + \left(\frac{3}{2}d\right)^2\right)$$

$$= \frac{1}{16}d^2(9+1+1+9) = \frac{20}{16}d^2 = \frac{5}{4}d^2$$

(b) Each symbol communicates $\log_2 M = \log_2 4 = 2$ bits.

Therefore, energy per bit $E_b = \frac{E_s}{2} = \frac{5}{8}d^2$

(c)



$$\hat{s}_{MAP}(r) = \begin{cases} s_1, & r < s_2 \\ s_2, & s_2 \leq r < s_3 \\ s_3, & s_3 \leq r < s_4 \\ s_4, & s_4 \leq r \end{cases} = \begin{cases} -\frac{3d}{2}, & r < -\frac{d}{2} \\ -\frac{d}{2}, & -\frac{d}{2} \leq r < \frac{d}{2} \\ \frac{d}{2}, & \frac{d}{2} \leq r < \frac{3d}{2} \\ \frac{3d}{2}, & \frac{3d}{2} \leq r \end{cases}$$

(d)

$$P(\mathcal{E} | S = s_i) = P[N > d] = e^{-\lambda d} \quad \text{for } i = 1, 2, 3$$

$$P(\mathcal{E} | S = s_4) = 0$$

$$P(\mathcal{E}) = \frac{3}{4} e^{-\lambda d}.$$

(e)

For comparable noise "power" with the Gaussian noise $N \sim \mathcal{N}(0, \sigma^2)$

we choose $\frac{1}{\lambda^2} = \sigma^2 \Rightarrow \frac{1}{\lambda} = \sigma$

mean of the exponential noise

mean of the exponential noise

From part (b), $E_b = \frac{5}{8} d^2$. So, $d = \sqrt{\frac{8}{5} E_b}$.

From part (d), $P(\varepsilon) = \frac{3}{4} e^{-\lambda d} = \frac{3}{4} \exp\left(-\frac{1}{\Delta}\sqrt{\frac{8}{5} E_b}\right) = \frac{3}{4} \exp\left(-2\sqrt{\frac{2}{5}\frac{E_b}{\Delta^2}}\right)$

This is the plot when the noise is Gaussian (which we derived in class).